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On Some Duality for Orthoposets

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We start an investigation of von Neumann semigroups. A connection between the variety of R-generated von Neumann semigroups and the category of orthoposets with dense morphisms is established.

1. INTRODUCTION

We would like to represent orthoposets by some suitable subclass of semigroups. Such attempts were established by Foulis (1960), Gudder (1972) Román [10,11] and Rumbos (1991), Román (1994) Zapatrin (1993), and others. The immediate motivation of our work was the representation of complete orthoposets via the category of involutive quantales contained in Mulvey and Pelletier (1992).

We have been also influenced by Pelletier and Rosický (1996), where the investigation of simple involutive quantales is developed. Finally, we should mention some other related papers (Borceux *et al.*, 1989; Lambek, 1995; Rosenthal, 1990) for the connection to the quantum and linear logic. For additional information concerning orthostructures see Kalmbach (1983).

In Section 2 we show the relation between the category of von Neumann semigroups and the category of orthoposets. It is noted that R-generated von Neumann semigroups form a variety. Similarly, in Section 3 the connection between the category of von Neumann semilattices and the category of ortholattices is discussed.

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2. VON NEUMANN SEMIGROUPS

Definition 2.1. An algebra $\mathcal{S} = (S, 0, 1, *, ^{\perp}, F, \cdot)$ with a signature (0, 0, 1, 1, 1, 2) where F is a discriminator function defined as

$$F(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 1 & \text{otherwise} \end{cases}$$

is called a *von Neumann semigroup* if it satisfies the following:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \tag{A}$$

$$0 \cdot a = 0 = a \cdot 0 \tag{N}$$

$$1 \cdot 1 = 1 \tag{T}$$

$$1 \cdot a \cdot 1 = F(a) \tag{Di}$$

$$a^{**} = a \tag{I1}$$

$$(a \cdot b)^* = b^* \cdot a^* \tag{I2}$$

$$1^* = 1 \tag{I3}$$

$$a^{\perp\perp} = a \cdot 1 \tag{O1}$$

$$a^{\perp\perp\perp} = a^{\perp} \tag{O2}$$

$$0^{\perp} = 1 \tag{O3}$$

$$a^* \cdot a^{\perp} = 0 \tag{Re}$$

$$a \cdot ((a^* \cdot b^{\perp})^{\perp} \cdot (b^* \cdot a^{\perp})^{\perp}) = b \cdot ((a^* \cdot b^{\perp})^{\perp} \cdot (b^* \cdot a^{\perp})^{\perp}) \tag{As}$$

$$(a^* \cdot c^{\perp})^{\perp} \cdot ((a^* \cdot b^{\perp})^{\perp} \cdot (b^* \cdot c^{\perp})^{\perp}) = (a^* \cdot b^{\perp})^{\perp} \cdot (b^* \cdot c^{\perp})^{\perp} \tag{Tr}$$

$$(1 \cdot a \cdot b \cdot 1)^{\perp} \cdot (1 \cdot a \cdot b^{\perp} \cdot 1)^{\perp} = (1 \cdot a \cdot 1)^{\perp} \tag{Is}$$

We put $R(S) = \{a \in S\colon a = a \cdot 1\}$, $L(S) = \{a \in S\colon a = 1 \cdot a\}$, $T(S) = \{a \in S\colon a = a \cdot 1\}$. $H(S) = \{a \in S\colon a = a^*\}$. Evidently, $T(S) = \{0, 1\}$. An element a, $a \in R(S)$ [$a \in L(S)$, $a \in T(S)$, $a \in H(S)$] is said to be *right-sided* (*left-sided, two-sided, self-adjoint*). Evidently, $0, 1, x^* \cdot x \in H(S)$. Let $a, b \in R(S)$. Then $b^* \cdot a \in T(S)$, i.e., we have $a^* \cdot a = 0$ iff $a = 0$ and $a^* \cdot a = 1$ iff $a \neq 0$. A von Neumann semigroup S is said to be *R-generated* if S is the least von Neumann semigroup of S containing $R(S)$.

A *morphism of von Neumann semigroups* is a map $f\colon S_1 \to S_2$ preserving 0, 1, e, $*$, $\perp$, F and $\cdot$. Recall that $f(x) = 0$ implies $x = 0$. Namely, $f(F(x)) = F(f(x)) = 0$ implies $F(x) = 0$, i.e., $x = 0$. We shall denote by $v\mathcal{N}e\mathcal{S}gr$ the category of von Neumann semigroups.

2.1. Constructing Orthoposets from von Neumann Semigroups

Let S be a von Neumann semigroup. We put $\mathcal{R}(S) = (R(S), \leq, 0, 1, {}^{\perp})$, where $0_{R(S)} = 0_S$, $1_{R(S)} = 1_S$, ${}^{\perp_{R(S)}} = {}^{\perp_{S/R(S)}}$, $r \leq y$ iff $x^* \cdot y^{\perp_S} = 0$.

Recall that an *orthocomplementation* on a bounded poset P is a unary operation ${}^{\perp}$ on P satisfying the following:

1. If $x \leq y$, then $y^{\perp} \leq x^{\perp}$.
2. $x^{\perp\perp} = x^{\perp}$.
3. The supremum $x \vee x^{\perp}$ and the infimum $x \wedge x^{\perp}$ exist and the equations $x \vee x^{\perp} = 1$ and $x \wedge x^{\perp} = 0$ hold.

Note that a map f between ordered sets with a bottom element 0 is said to be *dense* if $f(x) = 0$ implies $x = 0$. We shall denote by *Ortho* the category of orthoposets, i.e., bounded posets satisfying conditions 1–3, morphism are dense isotone mappings preserving 0, 1, and ${}^{\perp}$.

First, let us prove that $\leq$ is an ordering on $R(S)$. Let $a,b,c \in R(S)$, i.e., they are right-sided elements. By (Re) we have that $a^* \cdot a^{\perp} = 0$, i.e., $a \leq a$. Let $a \leq b$, $b \leq a$. Then $a^* \cdot b^{\perp} = 0$, $b^* \cdot a^{\perp} = 0$. By (As) we have

$$a = a \cdot (1 \cdot 1) = a \cdot ((a^* \cdot b^{\perp})^{\perp} \cdot (b^* \cdot a^{\perp})^{\perp})$$
$$= b \cdot ((a^* b^{\perp})^{\perp} \cdot (b^* \cdot a^{\perp})^{\perp}) = b \cdot (1 \cdot 1) = b$$

Let $a \leq b$, $b \leq c$. Then $a^* \cdot b^{\perp} = 0$, $b^* \cdot c^{\perp} = 0$. By (Tr) we have

$$(a^* \cdot c^{\perp})^{\perp} = (a^* \cdot c^{\perp})^{\perp} \cdot 1 \cdot 1 = (a^* \cdot c^{\perp})^{\perp} \cdot ((a^* \cdot b^{\perp})^{\perp} \cdot (b^* \cdot c^{\perp})^{\perp})$$
$$= (a^* \cdot b^{\perp})^{\perp} \cdot (b^* \cdot c^{\perp})^{\perp} = 1 \cdot 1 = 1$$

Now, let us prove that ${}^{\perp}$ is an orthocomplementation on $R(S)$. Evidently, the property 2 is satisfied by (O1). Let $a \leq b$. Then $a^* \cdot b^{\perp} = 0$, i.e., by (I1) and (I2) $b^{\perp *} \cdot a = 0$, i.e., by (O1), $b^{\perp *} \cdot a^{\perp\perp} = 0$, i.e., $b^{\perp} \leq a^{\perp}$, i.e., the property 1 is satisfied. Now, let $x \leq b$, $b^{\perp}$. Then $x^* \cdot b^{\perp} = 0$, $x^* \cdot b = 0$. Put $a = x^*$. Then by (Is) $1 = 1 \cdot 1 = (1 \cdot a \cdot b \cdot 1)^{\perp} \cdot (1 \cdot a \cdot b^{\perp} \cdot 1)^{\perp} = (1 \cdot a \cdot 1)^{\perp}$, i.e., $0 = a = x$. The rest follows from property 1.

Note the following evident property: Let $a, b, c, d \in R(S)$ such that $a \leq b$, $c \leq d$. Then $a^* \cdot c \leq b^* \cdot d$. To prove it, assume that $0 = b^* \cdot d$. Then $a \leq b \leq d^{\perp} \leq c^{\perp}$, i.e., $0 = a^* \cdot c$.

Now, let f: $S_1 \to S_2$ be a morphism of von Neumann semigroups. We shall define a morphism $\mathcal{R}(f)$: $R(S_1) \to R(S_2)$ of orthoposets by $\mathcal{R}(f) = f_{/R(S_1)}$. Since f preserves right-sided elements, the definition is correct. We shall prove that $\mathcal{R}(f)$ preserves 0, 1, $\leq$ and ${}^{\perp}$ and that $\mathcal{R}(f)$ is dense. It is enough to check that $\mathcal{R}(f)$ preserves the ordering, the rest is evident. Now, let $a \leq b$. Then $a^* \cdot b^{\perp} = 0$, i.e., $f(a)^* \cdot f(b)^{\perp} = f(a^* \cdot b^{\perp}) = f(0) = 0$, i.e., $f(a) \leq f(b)$.

So we have constructed a functor $\mathcal{R}$: $v\mathcal{N}e\mathcal{S}gr \to Ortho$.

2.2 Constructing von Neumann Semigroups from Orthoposets

Lemma 2.2. Let P be an orthoposet. Then the poset

$$Q(P) = \{(\varphi, \psi): \varphi: P^{op} \to P^{op} \text{ is a right adjoint to } \psi: P \to P\}$$

is an ordered von Neumann semigroup.

Proof. The induced order is given by the pointwise ordering of the mappings in P^{op} and P, respectively; the multiplication is defined by $(\varphi_1,\psi_1) \cdot (\varphi_2,\psi_2) := (\varphi_2 \circ \varphi_1, \psi_1 \circ \psi_2)$. Evidently, $\cdot$ is an associative operation.

Similarly as in Mulvey and Pelletier (1992), we define two mappings κ_b, ${}_s\lambda$: $P \to P, b, s \in P$, by

$$\kappa_b(a) = \begin{cases} b & \text{if} \quad a \neq 0 \\ 0 & \text{if} \quad a = 0 \end{cases} \qquad {}_s\lambda(a) = \begin{cases} 1 & \text{if} \quad a \geq s \\ 0 & \text{otherwise} \end{cases}$$

Note that the right adjoint to κ_b is ${}_b\lambda$. Analogously, we shall define two mappings λ_s, ${}_b\kappa$: $P \to P, b, s \in P$, by

$${}_b\kappa(a) = \begin{cases} b & \text{if} \quad a \neq 1 \\ 1 & \text{if} \quad a = 1 \end{cases} \qquad \lambda_s(a) = \begin{cases} 0 & \text{if} \quad a \leq s \\ 1 & \text{otherwise} \end{cases}$$

We have that the right adjoint to λ_b is the map ${}_b\kappa$.

Recall that the bottom element 0 of $Q(P)$ has the form $({}_0\lambda, \kappa_0) = ({}_1\kappa, \lambda_1)$ and the top element 1 has the form $({}_1\lambda, \kappa_1) = ({}_0\kappa, \lambda_0)$.

Now, let us describe the right-sided elements of $Q(P)$. We have $(\varphi, \psi) \cdot ({}_1\lambda, \kappa_1) = (\varphi, \psi)$ iff $\psi(1) = \psi(a)$ for all $a \in P - \{0\}$ and $\varphi(c) = 1$ for all $c \geq s$ for some $s \in P$. Then $(\varphi, \psi) \in R(Q(P))$ iff $(\varphi, \psi) = ({}_b\lambda, \kappa_b)$ for some element $b \in P$ $[b = \psi(1)]$. Note that $\vee\kappa_{b_i} = \kappa_{\vee b_i}$, $\wedge\kappa_{b_i} = \kappa_{\wedge b_i}$, ${}_{\wedge s_i}\lambda = {}_{\vee s_i}\lambda$, ${}_{\vee s_i}\lambda = {}_{\wedge s_i}\lambda$ if the respective suprema and infima exist.

Similarly, for left-sided elements of $Q(P)$ we have $({}_1\lambda, \kappa_1) \cdot (\varphi, \psi) = (\varphi, \psi)$ iff $\psi(a) = 0$ for all $a \leq s$ and $\varphi(1_{P^{op}}) = \varphi(c)$ for all $c \neq 0_{P^{op}}$. Then $(\varphi, \psi) \in L(Q(P))$ iff $(\varphi, \psi) = ({}_d\kappa, \lambda_d)$ for some elements of $d \in P$ $[d = \varphi(0)]$. Recall that $\vee \lambda_{s_i} = \lambda_{\wedge s_i}$, $\wedge \lambda_{s_i} = \lambda_{\vee b_i}$, ${}_{\wedge b_i}\kappa = {}_{\wedge b_i}\kappa$, ${}_{\vee b_i}\kappa = {}_{\vee b_i}\kappa$ if the respective suprema and infima exist.

Now, let $(\varphi,\psi) \in T(Q(P))$. Then $(\varphi, \psi) = ({}_h\kappa, \lambda_a) = ({}_b\lambda, \kappa_b)$ for suitable elements $a, b \in P$. But this gives us that ($a = 0$ and $b = 1$) or ($a = 1$ and $b = 0$), i.e., we have exactly two two-sided elements of $Q(P)$, i.e., $T(Q(P)) = \{({}_b, \lambda, \kappa_0), ({}_1\lambda, \kappa_1)\}$.

Note that $(\varphi, \psi) \circ ({}_1\lambda, \kappa_1) = ({}_{\psi(1)}\lambda, \kappa_{\psi(1)})$ and $({}_1\lambda, \kappa_1) \circ (\varphi, \psi) = ({}_{\varphi(0)}\kappa, \lambda_{\varphi(0)})$.

The involution * is defined by $(\varphi, \psi)^* := ({}^\perp \circ \psi \circ {}^\perp, {}^\perp \circ \varphi \circ {}^\perp)$, the operation $^\perp$ is given by $(\varphi, \psi)_\perp := (\psi(1)^{\perp\lambda}, \kappa_{\psi(1)\perp})$, and the operation F is defined by (Di). It is a technical task to check that our operations satisfy the axioms of von Neumann semigroups. ■

Let $N(P)$ be the least von Neumann semigroup of $Q(P)$ generated by $\mathcal{R}(P)$, $N(f)$ a morphism of von Neumann semigroups from $N(P_1)$ to $N(P_2)$ such that $N(f)({}_b\lambda, \kappa_b) = ({}_{f(b)}\lambda, \kappa_{f(b)})$, f: $P_1 \to P_2$ being a dense morphism of orthoposets. Evidently, we have a functor $\mathcal{N}$: *Ortho* $\to$ *vNeSgr.*

Recall that it is easy to show that any R-generated von Neumann semigroup $\mathcal{S}$ satisfies the identity

$$a \cdot 1 \cdot a = a \tag{RL}$$

i.e., any element $x \in \mathcal{S}$ may be written as $x_R \cdot x_L$, $x_R \in R(S)$, $x_L \in L(S)$ and the category of R-generated semigroups is in fact a variety.

Theorem 2.3. Representation theorem for orthoposets. $\mathcal{R}$: *vNeSgr* $\to$ *Ortho* is a right adjoint to $\mathcal{N}$: *Ortho* $\to$ *vNeSgr.* Moreover, this adjunction determines a reflective subcategory of R-generated von Neumann semigroups and the variety of R-generated von Neumann semigroups is equivalent to the category of orthoposets with dense morphisms.

3. VON NEUMANN SEMILATTICES

Definition 3.1. An algebra $\mathcal{N} = (S, 0, 1, *, {}^\perp, F, \cdot\ \vee)$ such that $(S, 0, 1, *, {}^\perp, F, \cdot)$ is a von Neumann semigroup, $(S, 0, 1, \vee)$ is a $\vee$-semilattice with the top element 1 and the bottom element 0, i.e., the following identities hold:

$$a \vee b = b \vee a \tag{S1}$$

$$(a \vee b) \vee c = a \vee (b \vee c) \tag{S2}$$

$$a \vee 0 = a \tag{S3}$$

$$a \vee 1 = 1 \tag{S4}$$

and the identities

$$a \cdot (b \vee c) = a \cdot b \vee a \cdot c \tag{J1}$$

$$(b \vee c) \cdot a = b \cdot a \vee c \cdot a \tag{J2}$$

$$a^* \cdot ((a \vee b) \cdot 1)^\perp = 0 \tag{J3}$$

hold, is called a *von Neumann semilattice.*

A von Neumann semilattice S is said to be *R-generated* if S is the least von Neumann semilattice of S containing $R(S)$.

A *morphism of von Neumann semilattices* is both a lattice morphism and a morphism of von Neumann semigroups. We shall denote by $v\mathcal{N}e\mathcal{S}em$ the category of von Neumann semilattices.

3.1. Constructing Ortholattices from von Neumann Semilattices

Let S be a von Neumann semilattice. We put $\mathcal{R}_1(S) = (R(S), 0, 1, {}^{\perp}, \vee, \wedge)$, where $0_{R(S)} = 0_S$, $1_{R(S)} = 1_S$, ${}^{\perp_{m(s)}} = {}^{\perp^{s}}$, $x \vee_{R(S)} y = x \vee_S$. Since finite suprema preserve right-sideness, our definitions are correct.

First, let us show that the lattice ordering $\leq_{lot}$ and the ordering $\leq_{pos}$ from Section 2.1 coincide. Evidently, for all $x,y \in R(S)$, $x \leq_{pos} x \vee y$, $y \leq_{pos} x \wedge y$ by (J3). Now, let z be any upper bound of x, y, i.e., $x \leq_{pos} z$, i.e., $x^* \cdot z = 0$, $y^* \cdot z = 0$, i.e., $(x^* \vee y^*) \cdot z = 0$, i.e., $x \vee y \leq_{pos} z$, i.e., the poset $R(S)$ is a lattice and its lattice ordering coincides with the lattice ordering from S.

We shall denote by *Ortholatt* the category of ortholattices, i.e., orthoposets that are lattices, morphism are dense lattice homomorphisms preserving ${}^{\perp}$. Evidently, *Ortholatt* is a subcategory of *Ortho*.

Now, let f: $S_1 \to S_2$, be a morphism of von Neumann semilattices. We shall define a dense morphism $\mathcal{R}_1(f){:}R(S_1) \to R(S_2)$ of ortholattices by $\mathcal{R}_1(f) = f_{/R(S_1)}$. Since f preserves finite suprema, the definition is correct. We have constructed a functor of $\mathcal{R}_t$: $v\mathcal{N}e\mathcal{S}em \to$ *Ortholatt.*

3.2. Constructing von Neumann Semilatticess from Ortholattices

Lemma 3.2. Let $P = (P, 0, 1, {}^{\perp}, \vee, \wedge)$ be an ortholattice. Then $Q(P)$ is a von Neumann semilattice.

Proof. Since for any lattice the join of two left adjoints is a left adjoint, the finite joins in $Q(P)$ are defined pointwise in P^{op} and P, respectively, i.e., we have $(\varphi_t, \psi_1) \vee (\varphi_2, \psi_2)\ (s,t) := (\varphi_1(s) \vee_{P^{op}} \varphi_2(s), \psi_1\ (t) \vee \psi_2(t))$. Evidently, $\vee$ satisfies (S1)–(S4) and (J1)–(J3). ■

Let $N_l(P)$ be the least von Neumann semilattice of $Q(P)$ generated by $\mathcal{R}(P)$, $N_l(f)$ a morphism of von Neumann semigroups from $N_i(P_1)$ to $N_i(P_2)$ such that $N_l(f)_{N(\mathrm{P})} = N(f), f{:}P_1 \to P_2$, being a dense morphism of ortholattices. We have a functor N_l : *Ortho* $\to$ $\upsilon NeSqr.$

Theorem 3.3 (Representation theorem for ortholattices). The functor $\mathcal{R}_1$: *vNeSem* $\to$ *Ortholatt* is a right adjoint to $\mathcal{N}_1$: *Ortholatt* $\to$ *vNeSem.* Moreover, this adjunction determines a reflective subcategory of R-generated von Neumann semilattices, and R-generated von Neumann semilattices correspond to ortholattices.

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